Variational principle for limit cycles of the Rayleigh–van der Pol equation

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We show that the amplitude of the limit cycle of Rayleigh's equation can be obtained from a variational principle. We use this principle to reobtain the asymptotic values for the period and amplitude of the Rayleigh and van der Pol equations. Limit cycles of general Liénard systems can also be derived from a variational principle. [S1063-651X(99)02705-1]

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I. INTRODUCTION

Limit cycles or self-excited oscillations appear in a wide class of problems in electronics, biology, astrophysics, to name a few. The first and most studied examples of equations which exhibit limit cycles are the Rayleigh equation

$$\ddot{y}+\nu\left(\frac{1}{3}\dot{y}^3-\dot{y}\right)+y=0, \tag{1}$$

which was introduced to show the appearance of sustained vibrations in acoustics [1], and the van der Pol equation [2]

$$\ddot{x}+\nu(x^2-1)\dot{x}+x=0, \tag{2}$$

derived for a certain electrical circuit. These systems exhibit a single periodic solution of period and amplitude depending on ν. It was soon noticed that these two equations are closely related. Indeed, taking the derivative of Rayleigh's equation and calling $\dot{y}=x$, we obtain van der Pol's equation for x. Therefore, the limit cycles for both equations have the same period and the amplitude $x_{\max}$ of the limit cycle of the van der Pol equation is the maximum value of $\dot{y}$. More general systems of the form

$$\ddot{x}+\nu f(x)\dot{x}+x=0 \tag{3}$$

were studied by Liénard [3], who gave conditions on f for the existence of a unique limit cycle. The conditions for the existence and uniqueness of limit cycles for systems of the form $\ddot{x}+\nu f(x)\dot{x}+g(x)=0$, or generalized Liénard systems, have also been established [4]. For functions $f(x)$ which do not satisfy Liénard's conditions, the existence, number, and location of limit cycles is an open problem. For small departures of the Hamiltonian case $\ddot{x}+x=0$ these questions can be answered by Melnikov's theory [5,6]. A new, nonperturbative, approach to solving this problem has been proposed recently [7,8].

In this paper we will be concerned with the Rayleigh–van der Pol equation. There is a large amount of work on this equation, both rigorous and numerical. The period and amplitude have been determined mainly by perturbation theory for the small and large ν limits. Accurate methods [9] have been developed to obtain regular series for intermediate values of ν. The leading order results in the limit $\nu\to 0$ are the period $T=2\pi$ and the amplitude of the limit cycle $A=2$. In the limit of $\nu\to\infty$ the period is, in leading order, $T=3-2\ln(2)$ and the amplitude for the van der Pol equation is again $A=2$. Higher order corrections for both the small and large ν [10] limits have been obtained. Detailed results and additional references on these and other points can be found in [10–13], among other books.

The purpose of this work is to present an alternative approach by noticing that a variational principle can be formulated for Rayleigh's equation and also for general Liénard equations (3). We derive this variational principle and show how to recover from it the period and amplitude in the limiting regions of the parameter ν. By using this variational principle together with appropriate trial functions one can obtain the amplitude and the period of the limit cycle for arbitrary values of ν.

II. VARIATIONAL PRINCIPLE

We shall work on Rayleigh's equation and derive the amplitude and period for Rayleigh's equation. Given the connection between the two equations mentioned in the Introduction, one can also obtain the amplitude of the limit cycle for the van der Pol equation. We shall use a method employed previously to obtain variational principles for other *nonlinear eigenvalue problems* [15–17] (for a review see [14]). Due to the symmetry of Rayleigh's equation, the limit cycle extends between a minimum $x_{\min}=-A$ and a maximum $x_{\max}=A$. Moreover, in phase space, if the point $(\dot{y},y)$ belongs to the limit cycle, then the point $(-\dot{y},-y)$ also belongs to it. Therefore we may consider the positive upper half $\dot{y}>0$ of the phase plane, where half a period will elapse when going from the points $(\dot{y}=0,x_{\min})$ to $(\dot{y}=0,x_{\max})$. Then the equation for the limit cycle in phase space can be written as the *nonlinear eigenvalue problem*,

$$p\frac{dp}{dy}+\nu\left(\frac{p^3}{3}-p\right)+y=0 \quad \text{with} \quad p(\pm A)=0 \quad \text{and} \quad p>0,$$

where we have called $p(y)=\dot{y}(y)$. The *nonlinear eigenvalue* is the amplitude A which appears in the boundary conditions. It is convenient to define a new variable $u=y/A$ in terms of which the equation for p is given by

$$\frac{1}{S}p\frac{dp}{du}+\left(\frac{p^3}{3}-p\right)+Ru=0, \tag{4}$$

with $p(\pm 1)=0$ and $p>0$.

Two parameters appear naturally in this problem, $R \equiv A/\nu$ and $S \equiv \nu A$. In terms of the function $p(u)$, the period of the limit cycle is given by

$$T = 2A \int_{-1}^{1} \frac{du}{p}.$$

Having written the problem in phase space, we may adapt a method to obtain variational principles developed for other problems. The method is simple enough that the full detail can be given here. Let $g(u)$ be a positive (continuous, with continuous derivative) but otherwise arbitrary function. Multiplying Eq. (4) by $g(u)$ and integrating in u between 0 and 1, we obtain, after integrating by parts and making use of the boundary conditions on p,

$$R \int_{-1}^{1} u g(u) du = \int_{-1}^{1} \phi(g(u), p(u)) du, \tag{5}$$

where

$$\phi(g(u), p(u)) = gp + \frac{1}{2S} p^2 g' - \frac{1}{3} g p^3$$

$$\equiv g\left(p + \frac{1}{2} p^2 v - \frac{1}{3} p^3 \right).$$

Here we have defined $v(u) \equiv g'(u)/(S g(u))$. At each value of u we may regard ϕ as a function of p; since g is positive, ϕ has a single maximum at a positive value $\hat{p}$ given by

$$\hat{p} = \frac{1}{2} [v + \sqrt{v^2 + 4}]. \tag{6}$$

Therefore $\phi(g,p) \leqslant \phi(g,\hat{p})$. Replacing this in the identity Eq. (5), we obtain

$$R \leqslant \frac{1}{12} \frac{\int_{-1}^{1} g(u) F(v(u)) du}{\int_{-1}^{1} u g(u) du}, \tag{7}$$

where $F(v) = 2v + (4 + v^2)[v + \sqrt{4 + v^2}]$. Therefore, given a trial function $g(u)$ and a value of S, through Eq. (7) we obtain a bound on R; the original parameters A and ν are given in terms of S and R by

$$A = (RS)^{1/2} \quad \text{and} \quad \nu = \left(\frac{S}{R} \right)^{1/2}.$$

The equality in Eq. (7) will hold whenever the trial function $g(u)$ is such that $\hat{p}$ is exactly p, the true solution for the limit cycle. From Eq. (6) we see that this will occur for $g = \hat{g}$ satisfying

$$\frac{1}{S} \frac{g'}{g} = v = p - \frac{1}{p},$$

from which it follows that

$$\hat{g}(u) = \exp\left(S \int_{-1}^{u} v(t) dt \right) = \exp\left[S \int_{-1}^{u} \left(p - \frac{1}{p} \right) dt \right], \tag{8}$$

which is a nonsingular positive function. Notice that the minimizing g is uniquely defined modulo a multiplicative constant. Our main result is then

$$R = \min_{g} \frac{1}{12} \frac{\int_{-1}^{1} g(u) F(v(u)) du}{\int_{-1}^{1} u g(u) du}, \tag{9}$$

where the minimization is over all positive functions $g(u)$ (which are continuous and have continuous derivative in [0,1]). Alternatively, one may express g and v in terms of $\hat{p}$, and consider the more intuitive function $\hat{p}$ as the trial function. In terms of $\hat{p}$ the above results read

$$R = \min_{\hat{p}} \frac{1}{6} \frac{\int_{-1}^{1} e^{S \int_{-1}^{u} [\hat{p} - (1/\hat{p})] dt} [\hat{p}^3 + 3\hat{p}] du}{\int_{-1}^{1} u e^{S \int_{-1}^{u} [\hat{p} - (1/\hat{p})] dt} du}.$$

To avoid confusion between the true solution of the equation $p(u)$ and the trial functions $\hat{p}(u)$, we will continue with the use of $g(u)$ (and also v), the original trial function. Notice also that we shall be interested in the maximum value of $v(u)$, which gives the maximum of $\hat{p}$, which in turn is the amplitude of the limit cycle of the van der Pol equation. For each trial function g we obtain a value of $\hat{p}$ and the corresponding approximate period is given by

$$T \approx 4A \int_{-1}^{1} \frac{du}{v + \sqrt{v^2 + 4}}. \tag{10}$$

If we wish to obtain approximate values for the amplitude and period, we may choose different trial functions and optimize the bound numerically. Instead, we shall study analytically the large and small S limits to reobtain the known results in these limits.

III. ASYMPTOTIC RESULTS

In this section we reobtain the known results in the limits of small and large ν. Notice that the variational principle (9) is of the form $R = \min I[g,g']/J[g,g']$ with $I \equiv \int_{-1}^{1} g(u) F(v(u)) du/12$ and $J \equiv \int_{-1}^{1} u g(u) du$. In order to minimize the above quotient, it is better to introduce a Lagrange multiplier, and extremize instead the quantity

$$I(g,g') - \lambda J(g,g') = \int_{-1}^{1} L(w, w', u) du,$$

where λ is a Lagrange multiplier and where, rather than using variables g and g', we have introduced the variables

$$w(u) = \int_{-1}^{u} v(s) ds \quad \text{and} \quad w'(u) = v(u).$$

The Euler-Lagrange equation

$$\frac{d}{dt}\left(\frac{\partial L}{\partial w'}\right)-\frac{\partial L}{\partial w}=0,$$

for this problem, is

$$\frac{1}{S}F''(v)v'+vF'(v)-F(v)+\lambda u=0 \tag{11}$$

or, more explicitly,

$$\frac{3}{S}\frac{d}{du}[v^2+v\sqrt{v^2+4}]+2v^3+2(v^2-2)\sqrt{v^2+4}+\lambda u=0. \tag{12}$$

This equation for $v(u)$ written in terms of p is, as expected, the original Eq. (4) in phase space with $R=\lambda/12$.

A. Small S limit

Let us consider first the limit of small S. In order to do this, we first obtain the approximate solution $v(u)$ of Eq. (12) for small S, which will depend on λ.

For small S, Eq. (12) for v becomes

$$\frac{3}{S}\frac{d}{du}[v^2+v\sqrt{v^2+4}]+\lambda u=0$$

which can be integrated to obtain

$$v^2+v\sqrt{v^2+4}+\frac{\lambda S}{6}u^2=C,$$

where C is a constant. Noticing that the above equation can be written as $2\hat{p}^2-2+\lambda S u^2/6=C$ and that the boundary conditions on $\hat{p}$ are $\hat{p}(\pm 1)=0$, evaluating at $u=1$ (or -1) we obtain the value for the constant $C=-2+\lambda S/6$. Defining $\delta=\lambda S/12$ we have that for small S

$$v^2+v\sqrt{v^2+4}+2\delta(u^2-1)+2=0,$$

and finally

$$v(u)=\sqrt{\delta(1-u^2)}-\frac{1}{\sqrt{\delta(1-u^2)}}. \tag{13}$$

Thus, for small S in leading order we have

$$I=\frac{1}{12}\int_{-1}^{1}\exp\left(S\int_{-1}^{u}v(t)dt\right)F(v(u))du$$

$$\approx\frac{1}{12}\int_{-1}^{1}F(v(u))du$$

and

$$J=\int_{-1}^{1}\exp\left(S\int_{-1}^{u}v(t)dt\right)u\,du\approx S\int_{-1}^{1}u\int_{-1}^{u}v(t)dt.$$

Using the expression (13) for $v(u)$ we obtain

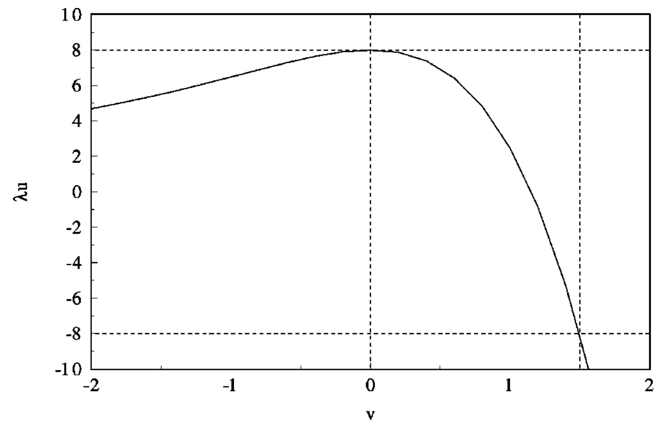

FIG. 1. Graph of the function $\lambda u(v)$.

$$I=\frac{3\pi}{4}\sqrt{\delta}(4+\delta),\quad J=\frac{3\pi S}{4\sqrt{\delta}}(3\delta-4),$$

so that from Eq. (7)

$$R\leqslant\frac{1}{S}\frac{\delta(4+\delta)}{3\delta-4}. \tag{14}$$

The right side of Eq. (14) is minimized at $\delta=4$, leading to the approximation $R\approx 4/S$ for small values of S. Therefore,

$$A=(RS)^{1/2}\approx 2,$$

which is the correct result in the small S limit. The period, evaluated from Eq. (10), with $v(u)$ given by Eq. (13), is $T=2\pi$.

B. Large S limit

Let us examine now the limit of large S. For large S, we obtain from Eq. (12)

$$\lambda u=-2v^3+2(2-v^2)\sqrt{v^2+4}. \tag{15}$$

To fix the value of λ we could proceed as above. However, the same results can be obtained in a more intuitive manner which we develop here. The function $\lambda u(v)$ is plotted in Fig. 1. Its maximum occurs at $v=0$ and its value is 8. Since the maximum value of u is 1 (recall that with our choice of variables u ranges from -1 to 1) we must have $\lambda=8$. With this choice for λ, the maximum value for v which occurs at $u=-1$ is 3/2. We have then our first conclusion, namely that the maximum of v at large S is 3/2 and the corresponding maximum for p is $(v_{\max}+\sqrt{v_{\max}^2+4})/2=2$. This maximum for p of the Rayleigh equation corresponds to the limiting amplitude of the van der Pol equation. The amplitude for the Rayleigh equation can be read from the variational principle. For large values of S the function $\hat{g}(u)$ given by Eq. (8) is highly peaked at the maximum of $\int_{-1}^{u}v(t)dt$, i.e., at $u=1$ (since in this case $v>0$). Therefore, for S large, from Eq. (9) we obtain

$$R\approx\frac{1}{12}F[v(1)]=\frac{1}{12}F[0]=\frac{2}{3},$$

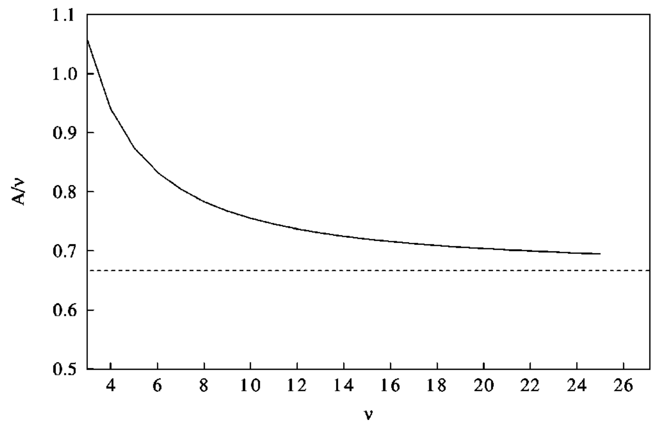

FIG. 2. Numerical results for the amplitude of the Rayleigh equation as a function of ν. The asymptotic behavior $A \to 2\nu/3$ can be noticed here.

and the asymptotic behavior for the amplitude is

$$A = \frac{2}{3}\nu. \tag{16}$$

In Fig. 2 we show the results of the numerical integration of Rayleigh's equation where we verify this asymptotic behavior. The period can be calculated from Eq. (10). In fact, by changing the independent variable from u to v, and by using Eq. (15), we obtain

$$T \approx 4A\int_{3/2}^{0} \frac{1}{v+\sqrt{v^2+4}}\frac{du}{dv}dv = 4A\left(\frac{9}{8} - \frac{3}{4}\operatorname{arcsinh}(3/4)\right). \tag{17}$$

Using Eq. (16) in Eq. (17) we finally obtain

$$T = [3 - 2\operatorname{arcsinh}(3/4)]\nu = (3 - 2\ln 2)\nu$$

which is the correct leading order for the period for large ν.

The two functions $v(u)$ which we have obtained as limiting values may also be used as convenient trial functions to obtain numerical bounds on R. In Fig. 3 we show the bound obtained using as a trial function the small S approximation for $v(u)$. With this, the simplest trial function, we obtain at $\nu = 4$ an error of less than 1.7%. Improvements of the maximum value of p, or equivalently on the amplitude of the van der Pol equation, can be obtained only by improving the trial function.

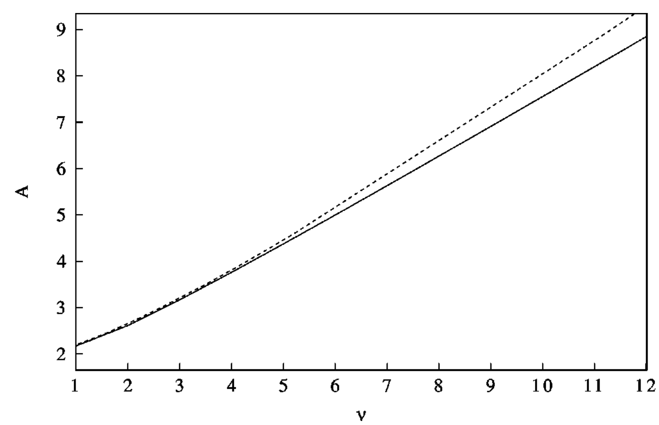

FIG. 3. Bounds on the amplitude of the limit cycle of the Rayleigh equation obtained with a simple trial function.

IV. CONCLUSION

We have shown that the equation satisfied by the limit cycle of Rayleigh's equation derives from the variational principle Eq. (9) which enables one to obtain the amplitude, in principle, with any desired accuracy. Once sufficient accuracy is obtained, the period and amplitude for the van der Pol equation follow from it. We have reobtained the limiting values for small and large deviations from the linear problem analytically. Our purpose in this work has been to show the existence of the variational principle, and the method of derivation. It is clear that a variational principle can also be formulated for Liénard systems of the form given in Eq. (3). The condition for the existence of a unique limit cycle are that f is even, $F(x) = \int_0^x f(s)ds$ satisfies $F < 0$ for $0 < x < x_0$, $F > 0$ for $x > x_0$, F is monotone increasing for $x > x_0$, and $F \to \infty$ as $x \to \infty$. These properties guarantee that the limit cycle of the associated equation $\ddot{y} + F(\dot{y}) + y = 0$ (obtained calling $x = \dot{y}$ and integrating) can be derived from a variational principle as well.

Several questions remain to be answered. First, for other types of functions f the equation may possess more than one limit cycle. It is direct to show that the phase space equation for the limit cycles of $\ddot{y} + F(\dot{y}) + y = 0$, for any polynomial F, can be derived from a variational principle. This means that all the limit cycles correspond to an extremum of a certain functional. It is an open question whether it is possible to count or estimate the positions of such extrema. Finally, the possibility of extending these results to generalized Liénard systems of the form $\ddot{x} + f(x)\dot{x} + g(x) = 0$ is another problem that remains open.

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